Problem 1.10

(a) How do the components of a vector transform under a **translation** of coordinates ($\overline{x} = x$, $\overline{y} = y - a$, $\overline{z} = z$, Fig. 1.16a)?

(b) How do the components of a vector transform under an **inversion** of coordinates ($\overline{x} = -x$, $\overline{y} = -y$, $\overline{z} = -z$, Fig. 1.16b)?

(c) How does the cross product (1.13) of two vectors transform under inversion? [The cross-product of two vectors is properly called a **pseudovector** because of this "anomalous" behavior.] Is the cross product of two pseudovectors a vector, or a pseudovector? Name two pseudovector quantities in classical mechanics.

(d) How does the scalar triple product of three vectors transform under inversions? (Such an object is called a **pseudoscalar.**)

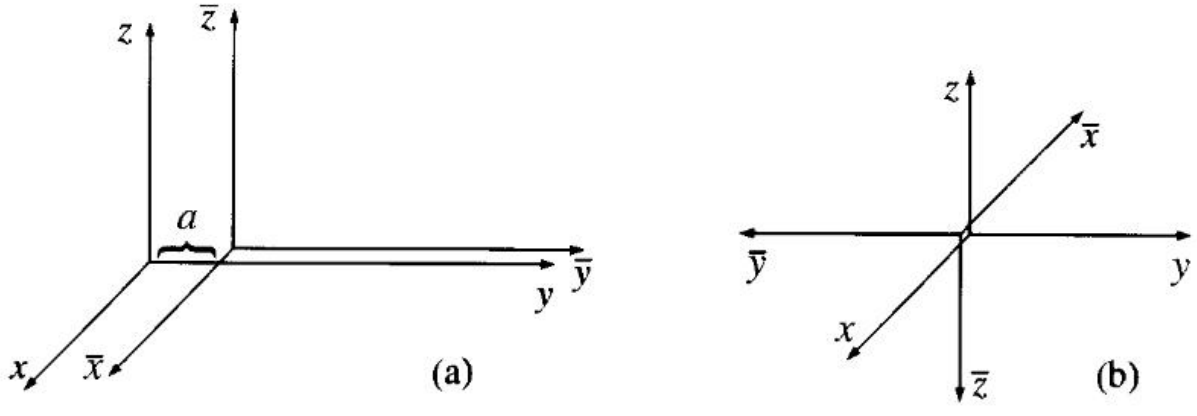


Figure 1.16

Problem 1.22 (For masochists only.) Prove product rules (ii) and (vi). Refer to Prob. 1.21 for the definition of $(\mathbf{A} \cdot \nabla)\mathbf{B}$.

Problem 1.37 Express the unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.64). Check your answers several ways ($\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \stackrel{?}{=} 1$, $\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} \stackrel{?}{=} 0$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \stackrel{?}{=} \hat{\boldsymbol{\phi}}$, ...). Also work out the inverse formulas, giving $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ (and θ, ϕ).

Problem 1.38

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

b) Do the same for $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob. 1.16.)

Problem 1.45

(a) Show that

$$x\frac{d}{dx}(\delta(x)) = -\delta(x).$$

[*Hint:* Use integration by parts.]

(b) Let $\theta(x)$ be the **step function**:

$$\theta(x) \equiv \left\{ \begin{array}{ll} 1, & \text{if } x > 0 \\ \\ 0, & \text{if } x \leq 0 \end{array} \right\}. \tag{1.95}$$

Show that $d\theta/dx = \delta(x)$.

Problem 1.48 Evaluate the integral

$$J = \int_{\mathcal{V}} e^{-r}\left(\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{r}}}{r^2}\right)d\tau$$

(where $\mathcal{V}$ is a sphere of radius R, centered at the origin) by two different methods, as in Ex. 1.16.

Problem 1.60 Although the gradient, divergence, and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show that:

(a) $\int_{\mathcal{V}}(\boldsymbol{\nabla}T)\,d\tau = \oint_{\mathcal{S}} T\,d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant, in the divergence theorem; use the product rules.]

(b) $\int_{\mathcal{V}}(\boldsymbol{\nabla}\times\mathbf{v})\,d\tau = -\oint_{\mathcal{S}}\mathbf{v}\times d\mathbf{a}$. [*Hint:* Replace $\mathbf{v}$ by $(\mathbf{v}\times\mathbf{c})$ in the divergence theorem.]

(c) $\int_{\mathcal{V}}[T\nabla^2 U + (\boldsymbol{\nabla}T)\cdot(\boldsymbol{\nabla}U)]\,d\tau = \oint_{\mathcal{S}}(T\boldsymbol{\nabla}U)\cdot d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = T\boldsymbol{\nabla}U$ in the divergence theorem.]

(d) $\int_{\mathcal{V}}(T\nabla^2 U - U\nabla^2 T)\,d\tau = \oint_{\mathcal{S}}(T\boldsymbol{\nabla}U - U\boldsymbol{\nabla}T)\cdot d\mathbf{a}$. [*Comment:* This is known as **Green's theorem**; it follows from (c), which is sometimes called **Green's identity**.]

(e) $\int_{\mathcal{S}}\boldsymbol{\nabla}T\times d\mathbf{a} = -\oint_{\mathcal{P}} T\,d\mathbf{l}$. [*Hint:* Let $\mathbf{v} = \mathbf{c}T$ in Stokes' theorem.]

Problem 1.61 The integral

$$\mathbf{a} \equiv \int_{\mathcal{S}} d\mathbf{a} \tag{1.106}$$

is sometimes called the **vector area** of the surface $\mathcal{S}$. If $\mathcal{S}$ happens to be *flat*, then $|\mathbf{a}|$ is the *ordinary* (scalar) area, obviously.

(a) Find the vector area of a hemispherical bowl of radius R.

(b) Show that $\mathbf{a} = 0$ for any *closed* surface. [*Hint:* Use Prob. 1.60a.]

(c) Show that $\mathbf{a}$ is the same for all surfaces sharing the same boundary.

(d) Show that

$$\mathbf{a} = \tfrac{1}{2} \oint \mathbf{r} \times d\mathbf{l}, \tag{1.107}$$

where the integral is around the boundary line. [*Hint:* One way to do it is to draw the cone subtended by the loop at the origin. Divide the conical surface up into infinitesimal triangular wedges, each with vertex at the origin and opposite side $d\mathbf{l}$, and exploit the geometrical interpretation of the cross product (Fig. 1.8).]

(e) Show that

$$\oint (\mathbf{c} \cdot \mathbf{r})\, d\mathbf{l} = \mathbf{a} \times \mathbf{c}, \tag{1.108}$$

for any constant vector $\mathbf{c}$. [*Hint:* let $T = \mathbf{c} \cdot \mathbf{r}$ in Prob. 1.60e.]