Chapter 8
Electrodynamics & Relativity

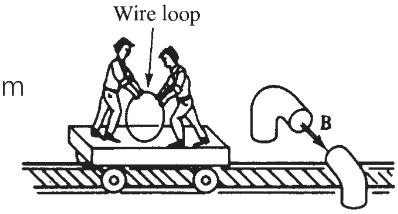
- Special Relativity
- Relativistic Mechanics
- Relativistic Electrodynamics

Special Relativity:

Does the classical relativity apply to electrodynamics?

A stationary point charge produces no B field.
A moving point charge produces B field.

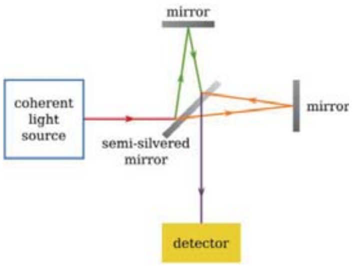
Where does the current come from in different references?



The need for a special reference frame (Ether)?

Test of Ether: Michelson-Morley experiment

The speed of light is the same in both directions!



There is no preferred reference frame!
-No Ether!

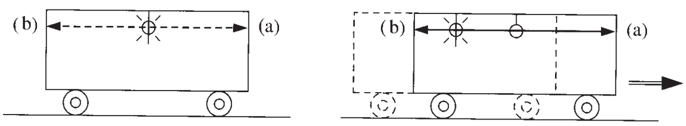
Einstein's two postulates for the special theory of relativity (STR):

1. The principle of relativity: The laws of physics apply in all inertial reference systems.

2. The universal speed of light: The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

It is a theory of inertial systems (without forces involved), that's why it's called a special theory.

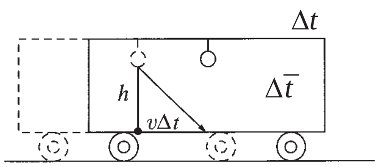
The relativity of simultaneity:



Two events that are simultaneous in one inertial system are not, in general, simultaneous in another.

A Event is something that happens at position x at time t, i.e., at (x, t).

Time Dilation:



The time from the car:

$$\Delta\bar{t} = \frac{h}{c}$$

The time on the ground:

$$\Delta t = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c} \Rightarrow \Delta t = \frac{h}{c}\frac{1}{\sqrt{1-(v/c)^2}} = \gamma\frac{h}{c}$$

So the time we measure in the two frames have the relation:

$$\Delta t = \gamma\Delta\bar{t}$$

Moving clocks run slow.

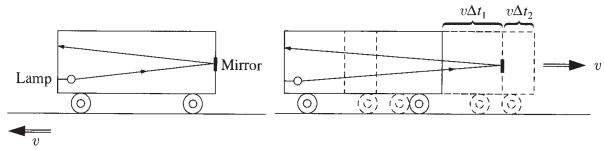
Time Dilation:

The time dilation is so strange to us. However, it has been confirmed in several experiments.

Muons are created by cosmic ray interaction with the upper atmosphere. At rest, their lifetime is about $2 \times 10^{-6} s$ and should not have time to reach the Earth's surface. The time dilation extends their life span as seen from Earth so they can be observed on Earth's surface.

You can also compare the lifetime of an accelerated particle with a rest one in lab.

Lorentz Contraction:



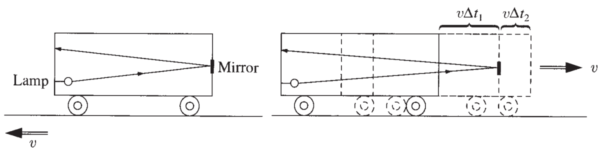
The time from the car: $\Delta \bar{t} = 2\frac{\Delta \bar{x}}{c}$

The time on the ground:

$$\Delta t_1 = \frac{\Delta x + v\Delta t_1}{c}, \quad \Delta t_2 = \frac{\Delta x - v\Delta t_2}{c}$$

$$\Rightarrow \Delta t_1 = \frac{\Delta x}{c-v}, \quad \Delta t_2 = \frac{\Delta x}{c+v}$$

Lorentz Contraction: $\Delta \bar{t} = 2\frac{\Delta \bar{x}}{c}$



The total time on the ground:

$$\Delta t = \Delta t_1 + \Delta t_2 = 2\frac{\Delta x}{c}\frac{1}{1-v^2/c^2} = 2\frac{\Delta x}{c}\gamma^2$$

According to time dilation: $\Delta t = \gamma \Delta \bar{t}$

We have the Lorentz Contraction: $\Delta x = \Delta \bar{x} / \gamma$

Moving objects are shortened.

Elements of STR:

- Two postulates.
- Time dilation.

  Moving clocks run slow.
- Lorentz contraction.

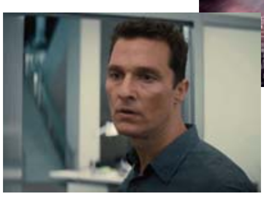
  Moving objects are shortened.

Dimensions perpendicular to the velocity are not contracted!

A example of time dilation from movie <Interstellar>

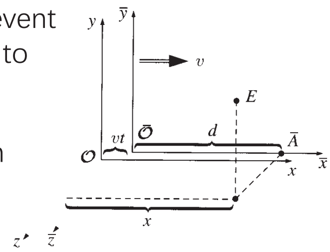
The dad and daughter:

After the dad finishes his travel:



The Lorentz Transformations:

We want to transform the event E from an inertial system S to another inertial system $\bar{S}$.



The Galilean transformation gives:

$$(i) \quad \bar{x} = x - vt$$

$$(ii) \quad \bar{y} = y$$

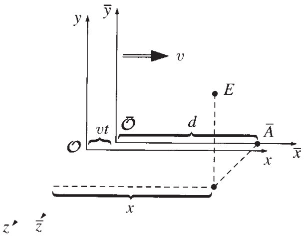
$$(iii) \quad \bar{z} = z$$

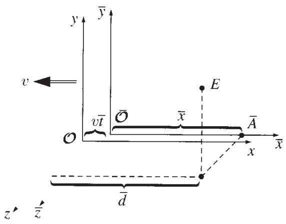
$$(iv) \quad \bar{t} = t$$

Is this still valid?

The Lorentz Transformations:

system S system $\bar{S}$





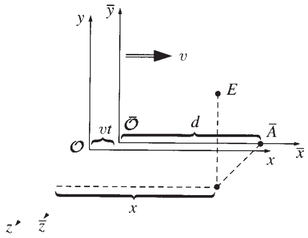
From Lorentz contraction: $d = x - vt = \frac{1}{\gamma}\bar{x}$

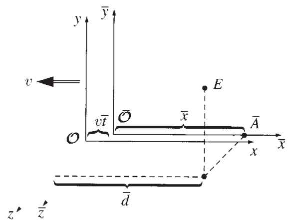
Again from:

$$\bar{x} = \bar{d} - v\bar{t} = x/\gamma - v\bar{t} \quad \Rightarrow \quad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$$

The Lorentz Transformations:

system S system $\bar{S}$





We arrive the Lorentz contraction:

$$(i)\quad \bar{x} = \gamma(x - vt) \qquad (ii)\quad \bar{y} = y$$

$$(iii)\quad \bar{z} = z \qquad (iv)\quad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$$

Structure of spacetime:

Define: $x^0 \equiv ct, \beta \equiv v/c, x^1 = x, x^2 = y, x^3 = z$

The Lorentz contraction becomes:

$$(i)\quad \bar{x} = \gamma(x - vt) \quad \Rightarrow \quad \bar{x}^1 = \gamma(x^1 - \beta x^0)$$

$$(ii)\quad \bar{y} = y \quad \Rightarrow \quad \bar{x}^2 = x^2$$

$$(iii)\quad \bar{z} = z \quad \Rightarrow \quad \bar{x}^3 = x^3$$

$$(iv)\quad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \quad \Rightarrow \quad \bar{x}^0 = \gamma(x^0 - \beta x^1)$$

Which can be written in matrix form:

$$\begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \quad \Rightarrow \quad \bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu$$

The vector form of Lorentz transformation is:

$$\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \equiv \Lambda^\mu_\nu x^\nu$$

The 4-vector:

$a^\mu = (a^0, a^1, a^2, a^3)$ is the contravariant vector.

$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$ covariant vector

We have:

$$a_\mu b^\mu = a^\mu b_\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

The displacement of two events A and B:

$$\Delta x^\mu = x_A^\mu - x_B^\mu$$

The interval is:

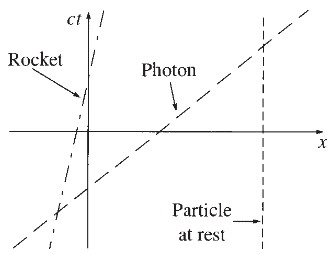
$$I \equiv \Delta x_\mu \Delta x^\mu = -c^2 t^2 + d^2$$

For $I < 0$, the interval is timelike.

For $I > 0$, the interval is spacelike.

For $I = 0$, the interval is lightlike.

Space-time diagrams:



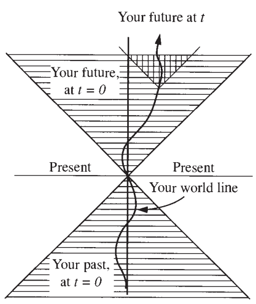
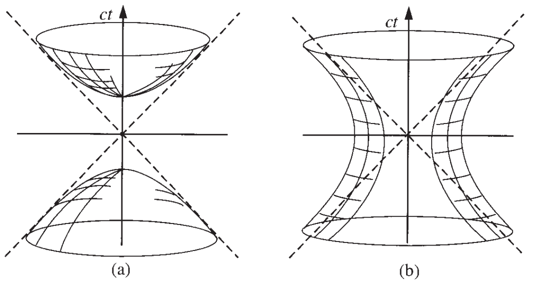


Figure 12.22

Space-time diagrams:



Timelike ($c^2t^2 > d^2$) Spacelike ($c^2t^2 < d^2$)

The invariant interval between causally related events is always timelike!

Can you solve the twin paradox?

Twin A stays on earth, while twin B travel at high speed to star X. When B returns, due to the time dilation, A find that B is younger than him.

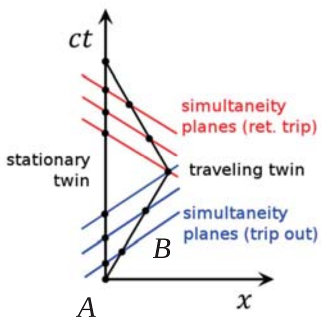
However, in B's reference frame, twin A is travelling at high speed. So when they meet, B think A is younger than him.

So which one is younger?

The space-time diagram for twin paradox:

It is not symmetric between A and B.

The changing of reference frame is the reason of this "discrepancy".



Let's check an example.

Ex: After their 21st birthday, twin B travel to star X and back immediately with speed $4c/5$. He returns at his 39th birthday.

1. How old is twin A? 51
2. How far is star X? 12

Let S, S', S'' be the reference frame of twin A, out trip of twin B, back trip of twin B. E is the event when B gets to star X.

Find coordinate of E in S, S', S''.

$(x, t) = (12,15)$

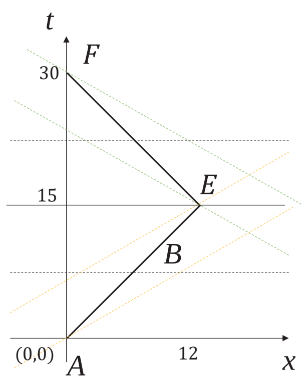
$(x', t') = (0,9)$ $(x'', t'') = (40,41)$

E:

S: $(x, t) = (12,15)$
S': $(x', t') = (0,9)$
S'': $(x'', t'') = (40,41)$

F:

S: $(x, t) = (0,30)$
S': $(x', t') = (-40,50)$
S'': $(x'', t'') = (40,50)$



So twin A's age according to B's return frame is:

$$21 + \frac{50}{\gamma} = 21 + 50 \cdot \frac{3}{5} = 51$$

Relativistic Mechanics: energy, momentum, and forces

Suppose you progress along your world line. Your proper (own) time $d\tau$ is connected to dt of ground observer due to time dilation:

$$dt = \frac{1}{\sqrt{1 - v^2 / c^2}} d\tau = \gamma d\tau$$

Proper time is independent of references.

Here velocity v is the relative speed of two inertial reference frames.

Now suppose you are flying to HK to join an important conference. The pilot announces the velocity is $4c/5$. What does he mean?

He means:

$$\vec{u} = \frac{d\vec{l}}{dt} = \frac{4}{5}c$$

dl and dt are both measured from the ground.

But you measure a different velocity:

$$\vec{\eta} = \frac{d\vec{l}}{d\tau} = \gamma\vec{u}$$

$d\tau$ is invariant.

Here u is called ordinary velocity, η is called proper velocity. The latter one transforms simply.

In 4-vector form: $\vec{\eta} = \frac{d\vec{l}}{d\tau} \Rightarrow \eta^\mu = \frac{dx^\mu}{d\tau}$

Since the proper time $d\tau$ is invariant, the Lorentz transformation from S to $\bar{S}$ gives:

$$\bar{\eta}^0 = \gamma\left(\eta^0 - \beta\eta^1\right)$$
$$\bar{\eta}^1 = \gamma\left(\eta^1 - \beta\eta^2\right)$$
$$\bar{\eta}^2 = \eta^2$$
$$\bar{\eta}^3 = \eta^3$$

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = c\gamma$$

Proper velocity 4-vector

$$\Longrightarrow \quad \bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu$$

This transformation is much simpler than ordinary velocities, where we need to transform both the denominator and numerator.

Relativistic Energy and Momentum:

We know $p = mv$. But in STR which v we need to use?

It turns out the relativistic momentum:

$$\vec{p} = m\vec{\eta} = m\gamma\vec{u}$$

$$p^\mu = m\eta^\mu \quad \Rightarrow \quad p^0 = m\eta^0 = m\frac{cdt}{d\tau} = \frac{mc}{\sqrt{1-u^2/c^2}}$$

The relativistic energy:

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} \quad \Longrightarrow \quad p^0 = E/c$$

Relativistic Energy and Momentum:

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

The rest energy: $E_{rest} = mc^2$

The kinetic energy:

$$E_{kin} = E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right)$$
$$\approx \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots$$

For $u \ll c$, we get the classical formula.

Is the energy invariant? Is the energy conserved?

invariant: same value in all inertial systems

conserved: same value before and after process

energy/rel mass is conserved, not invariant.
momentum is conserved, not invariant.
mass is invariant, not conserved.
velocity is neither conserved nor invariant.
charge is both conserved and invariant.

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = m_{rel}c^2$$

$$p^\mu p_\mu = -\left(p^0\right)^2 + \vec{p}\cdot\vec{p}$$
$$= -\frac{m^2c^4}{1-u^2/c^2}/c^2 + \frac{m^2u^2}{1-u^2/c^2}$$
$$= -m^2c^2$$

So we have the famous energy-momentum relation:

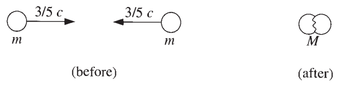
$$\boxed{E^2 - p^2c^2 = m^2c^4}$$

You can calculate E/p without knowing its velocity using this relation.

Relativistic Kinematics:

Ex: Two particles with mass m collide head-on at $3/5c$. They stick together. What is the mass (M) of the composite particle?

Sol:



$$\vec{p}_{before} = \vec{p}_{after} = 0$$

$$E_{before} = \frac{mc^2}{\sqrt{1-(3/5)^2}} \times 2 = \frac{5}{2}mc^2$$

$$E_{after} = Mc^2 = E_{before} = \frac{mc^2}{\sqrt{1-(3/5)^2}} \times 2 = \frac{5}{2}mc^2$$

$$\Longrightarrow \quad M = \frac{5}{2}m$$

Massless particle:

In classical mechanics there's no massless particle.

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

For massless particle, E and p go to zero.

However we know for a photon $m = 0$. But we have learned that EM field carries non-zero E and p. What's wrong?

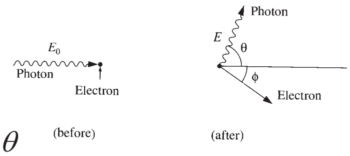
There is only one possibility:

The velocity of a photon is c. Photon has to travel at the speed of light!

$$E^2 - p^2c^2 = m^2c^4 \quad \Longrightarrow \quad E = pc = h\nu \quad \text{photon}$$

Compton Scattering:

A photon of energy E_0 bounces off an electron, initially at rest. Find the energy E as a function of scattering angle θ.



Conservation of p:

$$p_e \sin\phi = p_p \sin\theta$$

$$E_0/c = p_e\cos\phi + p_p\cos\theta$$

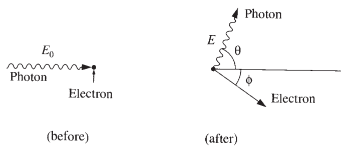
$$\Longrightarrow \quad p_e^2c^2 = E_0^2 - 2E_0E\cos\theta + E^2$$

Conservation of E:

$$E_0 + mc^2 = E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2}$$

$$= E + \sqrt{m^2c^4 + E_0^2 - 2E_0E\cos\theta + E^2}$$

Compton Scattering:



From the equation:

$$E_0 + mc^2 = E + \sqrt{m^2c^4 + E_0^2 - 2E_0E\cos\theta + E^2}$$

We get:

$$E = \frac{1}{(1-\cos\theta)/mc^2 + 1/E_0} = h\nu = \frac{hc}{\lambda}$$

$$\Longrightarrow \quad \lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)$$

$\hbar/mc$ is the Compton wavelength of the electron.

Relativistic Dynamics:

Newton's first law is built into the principle of relativity.

Newton's second law is valid in relativistic mechanics, when you use the relativistic momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}, \quad \vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

Is Newton's third law still valid in relativistic mechanics?

Work-Energy theorem:

The net work done on a particle equals the increase in its kinetic energy.

$$W = \int \vec{F}\cdot d\vec{l} = \int \frac{d\vec{p}}{dt}\cdot\frac{d\vec{l}}{dt}dt = \int \frac{d\vec{p}}{dt}\cdot\vec{u}dt$$

$$\frac{d\vec{p}}{dt}\cdot\vec{u} = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right)\cdot\vec{u} = \left(\frac{u/c^2}{(1-u^2/c^2)^{3/2}}\frac{du}{dt}m\vec{u} + \frac{1}{\sqrt{1-u^2/c^2}}\frac{md\vec{u}}{dt}\right)\cdot\vec{u}$$

$$= \frac{1}{(1-u^2/c^2)^{3/2}}\left[\frac{mu^3}{c^2}\frac{du}{dt} + \left(1-\frac{u^2}{c^2}\right)mu\frac{du}{dt}\right] = \frac{mu}{(1-u^2/c^2)^{3/2}}\frac{du}{dt}$$

$$= \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{dE}{dt}$$

$$\Longrightarrow \quad W = \int \frac{dE}{dt}dt = E_{final} - E_{initial}$$

Ordinary Force:

The transformation of F is ugly:

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma\left(dt - \beta dx / c\right)} = \frac{F_y}{\gamma\left(1 - \beta u_x / c\right)}$$

$$\bar{F}_z = \frac{d\bar{p}_z}{d\bar{t}} = \frac{dp_z}{\gamma\left(dt - \beta dx / c\right)} = \frac{F_z}{\gamma\left(1 - \beta u_x / c\right)}$$

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma\left(dp_x - \beta dp^0\right)}{\gamma\left(dt - \beta dx / c\right)} = \frac{F_x - \frac{\beta}{c}\frac{dE}{dt}}{1 - \beta u_x / c}$$

If $u = 0$, then:

$$\bar{F}_{\parallel} = F_{\parallel}, \quad \bar{F}_{\perp} = \frac{1}{\gamma} F_{\perp}$$

Minkowski Force:

You might have guessed there is a simplified force-Minkowski force, which transform simply:

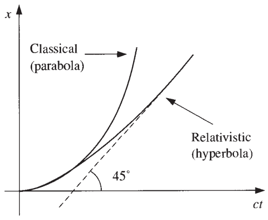
$$\vec{K} = \frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - u^2 / c^2}} \vec{F}$$

However, since the dynamics is dependent of ordinary time, the ordinary force is more useful sometime. For example, Lorentz's force law is a law of ordinary force:

$$\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right) \neq \vec{K}$$

Ex: A particle of mass m is subjected to a constant force F. It start from rest at origin. Find its position as a function of time.

Sol:



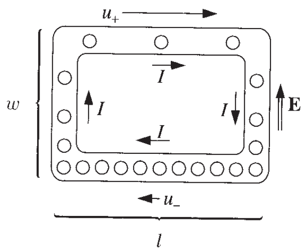
$$F = \frac{dp}{dt} \quad \Rightarrow \quad p = m\gamma u = Ft$$

$$u = \frac{Ft / m}{\sqrt{1 + \left(Ft / mc\right)^2}}$$

$$x(t) = \int_0^t u\left(t'\right) dt' = \int_0^t \frac{Ft' / m}{\sqrt{1 + \left(Ft' / mc\right)^2}} dt' = \frac{mc^2}{F}\left[\sqrt{1 + \left(Ft / mc\right)^2} - 1\right]$$

In classical limit: $\frac{Ft}{m} \ll c \qquad x(t) \to \frac{F}{2m} t^2$

Ex: As a model for a magnetic dipole m, a uniform electric field E accelerates the charges in the left while decelerates in the right. Find the total momentum.



Sol:

Because of the symmetry, the momentum on the left and right side cancel.

The current on the upper and lower side is the same:

$$I = \frac{QN_+}{l} u_+ = \frac{QN_-}{l} u_- \quad \Rightarrow \quad N_{\pm} u_{\pm} = \frac{Il}{Q}$$

N: # of particles $\qquad Q$: charge of each particle

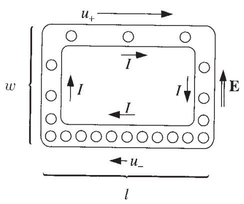
Classically we have:

$$p_{classical} = Mu_+N_+ - Mu_-N_- = 0$$

Relativistically we have:

$$p = \gamma Mu = \gamma_+ Mu_+N_+ - \gamma_- Mu_-N_-$$

$$= \frac{MIl}{Q}\left(\gamma_+ - \gamma_-\right)$$



$$N_{\pm} u_{\pm} = \frac{Il}{Q}$$

The energy gain from low to up side:

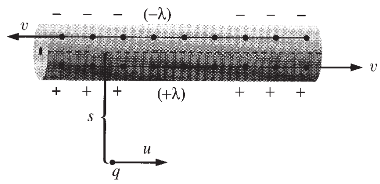
$$\left(\gamma_+ - \gamma_-\right) Mc^2 = QEw$$

So we get: $\vec{p} = \frac{IlEw}{c^2}\hat{p} = \frac{1}{c^2}\left(\vec{m} \times \vec{E}\right), \quad m = Ilw$

p to the right, m into the page

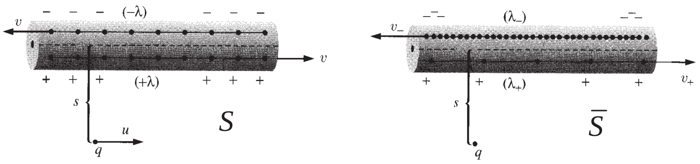
Relativistic Electrodynamics:

Now imagining you live on a planet where people know nothing about magnetism. But they know much about special relativity, however.



$$I = 2\lambda v$$

No net charges in the wire $\Rightarrow$ No forces on q

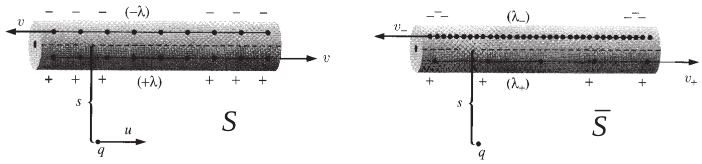


In the charge q frame $\bar{S}$: $v_\pm = \dfrac{v \mp u}{1 \mp vu/c^2}$

The charge density: $\lambda_\pm = \pm\gamma_\pm\lambda_0 = \pm\dfrac{1}{\sqrt{1-v_\pm^2/c^2}}\lambda_0$

In S frame: $\lambda = \gamma\lambda_0 = \lambda_0/\sqrt{1-v^2/c^2}$

We get: $\gamma_\pm = \gamma\dfrac{1 \mp uv/c^2}{\sqrt{1-u^2/c^2}}$



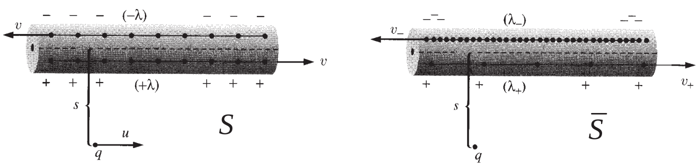
So the total charge density in $\bar{S}$:

$$\lambda_{tot} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda uv}{c^2\sqrt{1-u^2/c^2}}$$

This net charge sets up an electric field: $E = \dfrac{\lambda_{tot}}{2\pi\varepsilon_0 s}$

So there will be electric force $\bar{F}$ acting on q:

$$\bar{F} = qE = \frac{q\lambda_{tot}}{2\pi\varepsilon_0 s} = -\frac{\lambda v}{\pi\varepsilon_0 c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}$$



Since the physics is the same, there must be also a force on q in frame S:

$$F = \sqrt{1-u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\varepsilon_0 c^2}\frac{qu}{s} = -qu\left(\frac{\mu_0 I}{2\pi s}\right)$$

This means there is a hidden force F in S. This is exactly the Lorentz force! Then you call it a magnetic force and define the magnetic field:

$$B = \mu_0 I / 2\pi s$$

From above calculation we learn that:

Magnetism is a relativistic effect.

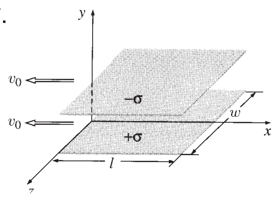
The origin of Magnetism comes from relativistic dynamics of charges, i.e., comes from the relativistic electrodynamics.

Why a charge current produces magnetic field?

Because the moving charges see different spacetime, leading to a hidden force to any point charge.

The interpretation may be different. Magnetic field in one frame might be electric in another.

Now we want to see what's the transformation rule from one frame to another.



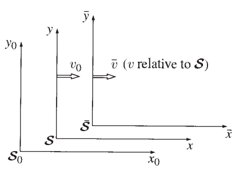
$$E_y = \frac{\sigma}{\varepsilon_0}, \quad B_z = -\mu_0\sigma v_0$$

Transform to $\bar{S}$:

$$\bar{E}_y = \frac{\bar{\sigma}}{\varepsilon_0}, \quad \bar{B}_z = -\mu_0\bar{\sigma}\bar{v}$$

$$\bar{v} = \frac{v+v_0}{1+vv_0/c^2}, \quad \bar{\gamma} = \frac{1}{\sqrt{1-\bar{v}^2/c^2}}$$

$$\bar{\sigma} = \bar{\gamma}\sigma_0$$



$$\Rightarrow \quad \bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\varepsilon_0}, \quad \bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0\sigma\bar{v}$$

Since $\dfrac{\bar{\gamma}}{\gamma_0} = \dfrac{\sqrt{1-v_0^2/c^2}}{\sqrt{1-\bar{v}^2/c^2}} = \dfrac{1+vv_0/c^2}{\sqrt{1-v^2/c^2}} = \gamma\left(1+\dfrac{vv_0}{c^2}\right)$

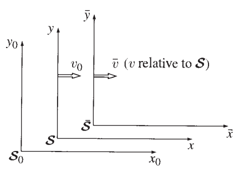
We have

$$\bar{E}_y = \gamma\left(1+\frac{vv_0}{c^2}\right)\frac{\sigma}{\varepsilon_0} = \gamma\left(E_y - \frac{v}{c^2\varepsilon_0\mu_0}B_z\right)$$

$$\bar{B}_z = -\gamma\left(1+\frac{vv_0}{c^2}\right)\mu_0\sigma\left(\frac{v+v_0}{1+vv_0/c^2}\right) = \gamma\left(B_z - \mu_0\varepsilon_0 vE_y\right)$$

$$\Rightarrow \quad \bar{E}_y = \gamma\left(E_y - vB_z\right)$$

$$\bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$



Similarly we have:

$$\bar{E}_x = E_x \quad \bar{E}_y = \gamma\left(E_y - vB_z\right) \quad \bar{E}_z = \gamma\left(E_z + vB_y\right)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

Let's see some special cases. $\vec{v} = v\hat{x}$

For $B = 0$ in S:

$$\vec{\bar{B}} = \gamma\frac{v}{c^2}\left(E_z\hat{y} - E_y\hat{z}\right) = \frac{v}{c^2}\left(\bar{E}_z\hat{y} - \bar{E}_y\hat{z}\right) = -\frac{1}{c^2}\left(\vec{v}\times\vec{\bar{E}}\right)$$

For $E = 0$ in S:

$$\vec{\bar{E}} = -\gamma v\left(B_z\hat{y} - B_y\hat{z}\right) = -v\left(\bar{B}_z\hat{y} - \bar{B}_y\hat{z}\right) = \vec{v}\times\vec{\bar{B}}$$

Before we go to tensor form of Maxwell's equations, let's define an antisymmetric field tensor:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Its dual tensor is: $E/c \to B, \quad B \to -E/c$

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix}$$

Maxwell's equations can be written in tensor form:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

Let's proof that!

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = \mu_0 J^0$$

$$= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}\nabla\cdot\vec{E} = \mu_0 c\rho$$

⇨ Gauss's law:

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = \mu_0 J^1$$

$$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \nabla\times\vec{B}\right)_x = \mu_0 \vec{J}_x$$

$$\frac{\partial F^{2\nu}}{\partial x^\nu} = \cdots, \frac{\partial F^{3\nu}}{\partial x^\nu} = \cdots$$

⇨ Ampere's law with Maxwell's correction:

$$-\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \nabla\times\vec{B} = \mu_0\vec{J}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$$\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} = 0$$

$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad ⇨ \quad \nabla\cdot\vec{B} = 0$$

$$\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = 0 \qquad \frac{\partial G^{2\nu}}{\partial x^\nu} = \cdots$$

$$= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = 0 \qquad \frac{\partial G^{3\nu}}{\partial x^\nu} = \cdots$$

⇨ Faraday's law:

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix}$$

The field tensor can also be expressed in terms of 4-4-vector potential:

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

Where the 4-vector potential:

$$A^\mu = \left[V/c, A_x, A_y, A_z\right]$$

You will find in the Lorentz gauge, the Maxwell's equations reduce to a single formula:

$$\Box^2 A^\mu = -\mu_0 J^\mu$$

This is the most elegant and simplest formulation of Maxwell's eqns.!

The End.