1.1 VECTOR ANALYSIS

In this course, we are dealing with vectors most of the time. A vector has multiple component, for example vector
\[ \vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}, \]
where \( \hat{x}, \hat{y}, \hat{z} \) are the unit vectors in Cartesian coordinate. We define the dot (\( \cdot \)) and cross (\( \times \)) product of two vectors
\[ \vec{a} \cdot \vec{b} = |a| |b| \cos \theta_{ab}, \]
\[ \vec{a} \times \vec{b} = |a| |b| \sin \theta_{ab} \hat{n}, \]
where \( \hat{n} \) is perpendicular to \( \vec{a} \) and \( \vec{b} \) with the direction given by the right-hand rule. So we have \( \hat{x} \cdot \hat{x} = 1 \), \( \hat{x} \times \hat{y} = \hat{z} \), etc. The calculation of arbitrary vectors can always be reduced to the calculation of unit vectors. So we have
\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z, \]
\[ \vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{x} + (A_z B_x - A_x B_z) \hat{y} + (A_x B_y - A_y B_x) \hat{z} \]
\[ = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \]

We see that we can always write \( \vec{A} \times \vec{B} \) as a determinant. One can prove the following triple product rules
\[ \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}), \]
\[ \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \vec{C} \cdot (\vec{A} \cdot \vec{C} - \vec{C} \cdot \vec{A}). \]

Next we define the gradient operator \( \nabla = \partial_x \hat{x} + \partial_y \hat{y} + \partial_z \hat{z} \). Note \( \nabla \) is not actually a vector. For example, \( (\nabla x) y \neq (\nabla y) x \). An operator is meaningless without acting on a function. One should proof the following product rules:
\[
\begin{align*}
\nabla (fg) &= f (\nabla g) + g (\nabla f) \quad (1) \\
\n\nabla \left( \vec{A} \cdot \vec{B} \right) &= \vec{A} \times \left( \nabla \times \vec{B} \right) + \vec{B} \times \left( \nabla \times \vec{A} \right) + \left( \vec{A} \cdot \nabla \right) \vec{B} + \left( \vec{B} \cdot \nabla \right) \vec{A} \\
\n\nabla \cdot (f \vec{A}) &= f (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f) \quad (2) \\
\n\nabla \cdot (\vec{A} \times \vec{B}) &= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \\
\n\nabla \times (f \vec{A}) &= f (\nabla \times \vec{A}) - \vec{A} \times (\nabla f) \quad (3) \\
\n\n\nabla \times (\vec{A} \times \vec{B}) &= (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})
\end{align*}
\]

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It’s obvious from Eq. (1-3) one has
\[ \nabla \left( \frac{f}{g} \right) = \frac{g (\nabla f) - f (\nabla g)}{g^2} , \]
\[ \nabla \cdot \left( \frac{\vec{A}}{g} \right) = \frac{g (\nabla \cdot \vec{A}) - \vec{A} \cdot (\nabla g)}{g^2} , \]
\[ \nabla \times \left( \frac{\vec{A}}{g} \right) = \frac{g (\nabla \times \vec{A}) + \vec{A} \times (\nabla g)}{g^2} . \]

For a vector field \( \vec{A} \), one has the Divergence, Curl, and the Laplacian of the field: \( \nabla \cdot \vec{A} \), \( \nabla \times \vec{A} \), and \( \nabla^2 \vec{A} \).

![Diagram showing the divergence and curl of a vector field](image)

\[ \vec{A} = (x, y, 0), \quad \nabla \cdot \vec{A} = 2 \]
\[ \vec{A} = (y, -x, 0), \quad \nabla \times \vec{A} = -2z \]

Figure 1. Illustration of the Divergence and Curl of field \( \vec{A} \) in \( xy \) plane.

### 1.2 INTEGRAL THEOREM

We will explore a little bit about the differential geometry and manifold here. Don’t be nervous, and we will see how to calculate. You are encouraged to grab a related math textbook afterwards. Now first I will show you the generalized Stokes’ theorem:

\[ \int_{\partial M} w = \int_{M} dw \]

Here \( M \) is a \( m \)-dimensional manifold (a manifold is a topological space that locally homeomorphic to Euclidean space) with its boundary \( \partial M \) and dimension \( m - 1 \). The exterior differential \( w \) is of \( (m - 1) \)-form, while the exterior derivative \( dw \) is of \( m \)-form. Basically, the generalized Stokes' theorem states that the integral of \( (m - 1) \)-form exterior differential \( w \) on the boundary of \( m \)-dimensional manifold \( M \) equals the integral of its exterior derivative on this manifold. The exterior differential consists of a series of wedge products \( \wedge \) which obeys anti-commutation relationship, for example

\[ dx \]
\[ dx \wedge dy = -dy \wedge dx \]
\[ dx \wedge dy \wedge dz = dy \wedge dz \wedge dx \]

and the exterior derivative is defined as

\[ d (f dx^1 \wedge dx^2 \wedge dx^3 \cdots dx^n) = \left( \sum_i \frac{\partial f}{\partial x^i} dx^i \right) \wedge dx^1 \wedge dx^2 \wedge dx^3 \cdots dx^n. \]
Thus we can see
\[ dx \wedge dx = 0, \ d(dx) = 0. \]

Now we are ready to do some calculations. For a closed surface integral
\[
\oint \mathbf{A} \cdot d\mathbf{S} = \int A_x dydz + A_y dzdx + A_z dxdy
\]
\[
= \int_{\partial M} A_x dy \wedge dz + A_y dz \wedge dx + A_z dx \wedge dy
\]
\[
= \int_M \left( \frac{\partial A_x}{\partial x} dy + \frac{\partial A_x}{\partial y} dz \right) dx \wedge dy + \left( \frac{\partial A_y}{\partial x} dx + \frac{\partial A_y}{\partial y} dz \right) dy \wedge dz + \left( \frac{\partial A_z}{\partial x} dx + \frac{\partial A_z}{\partial y} dz \right) dz \wedge dx
\]
\[
= \int_M \left( \frac{\partial A_x}{\partial x} dy - \frac{\partial A_x}{\partial y} dz \right) dxdy + \left( \frac{\partial A_y}{\partial x} dx - \frac{\partial A_y}{\partial y} dz \right) dydz + \left( \frac{\partial A_z}{\partial x} dx - \frac{\partial A_z}{\partial y} dz \right) dzdx
\]
\[
= \int_M \nabla \times \mathbf{A} \cdot d\mathbf{S}.
\]

this gives the Gauss’s theorem and the classical Stokes’ theorem is
\[
\oint \mathbf{A} \cdot d\mathbf{l} = \int A_x dx + A_y dy + A_z dz
\]
\[
= \int_{\partial M} A_x dy \wedge dz + A_y dz \wedge dx + A_z dx \wedge dy
\]
\[
= \int_M \left( \frac{\partial A_x}{\partial x} dy + \frac{\partial A_x}{\partial y} dz \right) dx \wedge dy + \left( \frac{\partial A_y}{\partial x} dx + \frac{\partial A_y}{\partial y} dz \right) dy \wedge dz + \left( \frac{\partial A_z}{\partial x} dx + \frac{\partial A_z}{\partial y} dz \right) dz \wedge dx
\]
\[
= \int_M \nabla \times \mathbf{A} \cdot d\mathbf{l}.
\]

Thus we recover the Gauss’s and classical Stokes’ theorem. The former converts the volume integral to a closed surface integral, while the latter converts the surface integral to a closed line integral.

1.3 SPHERICAL COORDINATE

The function \( f(x, y, z) \) can also be represented using the spherical coordinate, i.e., \( f(r, \theta, \phi) \). Now I will show you how to transform the gradient \( \nabla f \) to spherical coordinate.

First we note
\[
\nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}.
\]
Using the chain rule we have

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x},
\]

\[
\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y},
\]

\[
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z}.
\]

The unit vector has the relationship

\[
\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z},
\]

\[
\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z},
\]

\[
\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y},
\]

and the coordinate

\[
x = r \sin \theta \cos \phi,
\]

\[
y = r \sin \theta \sin \phi,
\]

\[
z = r \cos \theta.
\]

So we have

\[
\frac{\partial r}{\partial x} = \sin \theta \cos \phi, \quad \frac{\partial r}{\partial y} = \sin \theta \sin \phi, \quad \frac{\partial r}{\partial z} = \cos \theta,
\]

\[
\frac{\partial \theta}{\partial x} = \frac{\cos \theta \cos \phi}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta \sin \phi}{r}, \quad \frac{\partial \theta}{\partial z} = \frac{-\sin \theta}{r},
\]

\[
\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r \sin \theta}, \quad \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r \sin \theta}, \quad \frac{\partial \phi}{\partial z} = 0.
\]

Plug above equations into \( \nabla f \) we have

\[
\nabla f = \frac{\partial f}{\partial r} (\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z})
\]

\[
+ \frac{1}{r} \frac{\partial f}{\partial \theta} (\cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z})
\]

\[
+ \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} (-\sin \phi \hat{x} + \cos \phi \hat{y})
\]

\[
= \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}.
\]

The same technique can be used to get the Laplacian. The generalization to cylindrical system is straightforward. We have:
\[ \begin{align*}
\nabla f &= \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi} \\
\nabla^2 f &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}
\end{align*} \]

### 1.4 Delta Function

In electrodynamics we will deal with functions like \( \vec{v} = \hat{r} / r^2 \). Since the divergence \( \nabla \cdot \vec{v} = 0 \), you might think \( \int \nabla \cdot \vec{v} \, dV = 0 \). However, it is not. From Gauss’s theorem, we have

\[ \hat{\nabla} \cdot \vec{v} \, dV = \hat{\nabla} \cdot \vec{S} = \int \frac{\hat{r}}{r^2} r^2 \sin \theta d\theta d\phi = 4\pi. \]

The reason for this discrepancy is that the divergence at \( r = 0 \) is not well defined. Since the integral of \( \nabla \cdot \vec{v} \) is finite, the divergence must go to infinity at the origin. We then define a delta function

\[ \delta (x) = \begin{cases} 
\infty & x = 0 \\
0 & x \neq 0 
\end{cases} \]

with the property \( \int_{-\infty}^{\infty} \delta (x) \, dx = 1 \), and we have

\[ \nabla \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3 (\vec{r}) \]

where \( \delta^3 (\vec{r}) = \delta (x) \delta (y) \delta (z) \) is the 3D delta function. For delta function, we have the properties

\[ \int_{-\infty}^{\infty} f (x) \delta (x) \, dx = f (0) \]

\[ \delta (ax) = \frac{\delta (x)}{|a|} \]

Next we want to proof the following identity:

\[ \lim_{a \to \infty} \frac{\sin ax}{\pi x} = \delta (x) \]

First let’s check if the integral is equal to 1. Since

\[ \int_{-\infty}^{\infty} \frac{\sin ax}{\pi x} \, dx = \int_{-\infty}^{\infty} \frac{\sin x}{\pi x} \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ix} - e^{-ix}}{x} \, dx = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx \]

we need to calculate the last integral. Life is easier if we generalize the integral to complex plane. The contour integral

\[ \oint \frac{e^{iz}}{z} \, dz = \int_{C_0}^{C_1} \frac{e^{iz}}{z} \, dz + \int_{C_1}^{C_2} \frac{e^{iz}}{z} \, dz + \int_{C_2}^{C_3} \frac{e^{iz}}{z} \, dz + \int_{C_3}^{C_0} \frac{e^{iz}}{z} \, dz \]

Here the \( C_3 \) integral goes to zero, because

\[ \int_{C_3} \frac{e^{iz}}{z} \, dz = \int_{C_3} \frac{e^{ir(\cos \theta + i \sin \theta)}}{re^{i\theta}} \, dr e^{i\theta} \leq \int_0^\pi \left| \frac{e^{ir(\cos \theta + i \sin \theta)}}{re^{i\theta}} \right| \, |r e^{i\theta}| \, d\theta = \int_0^\pi e^{-r \sin \theta} \, d\theta \overset{r \to \infty}{\longrightarrow} 0. \]

The \( C_0 \) integral can be get using the residue theorem. We double the integral and form a closed contour (see the red dashed line). Residue theorem states that this closed integral is connected to the residue of the singularity inside,

\[ \int_{C_0} \frac{e^{iz}}{z} \, dz = \frac{1}{2} \oint \frac{e^{iz}}{z} \, dz = -\frac{1}{2} \times 2\pi i \times \text{Res} \left[ \frac{e^{iz}}{z} \right]_{z=0} = -\pi i. \]
Figure 3. Contour of the integral $\oint \frac{e^{iz}}{z} \, dz$

Since the closed contour $C_0 + C_1 + C_2 + C_3$ has no singularities within, this integral goes to zero. So we have

$$\oint \frac{e^{iz}}{z} \, dz = \oint_{C_0} \frac{e^{iz}}{z} \, dz + \int_{-\infty}^{-\epsilon} e^{ix} \, dx + \int_{\epsilon}^{\infty} e^{ix} \, dx = -\pi i + \int_{-\infty}^{-\epsilon} e^{ix} \, dx + \int_{\epsilon}^{\infty} e^{ix} \, dx = 0.$$  

As $\epsilon \to 0$, we have

$$\int_{-\infty}^{\infty} e^{ix} \, dx = \pi i.$$

So we have proofed the integral of $\lim_{a \to \infty} \sin ax / \pi x$ is equal to 1. As $a$ goes to infinity, this function oscillates fast and goes to infinity at $x = 0$. So we can approximate this function as zero at $x \neq 0$ and infinity at $x = 0$. \(\blacksquare\)